Seminar Lecture - Predicate Calculus

1. An **open sentence** is a 'proposition with a variable.' Open sentences are sometimes called *predicates*.
2. **Example:** $x = 2$; x is an elephant; etc.
3. Generic proposition: P; generic open sentence: $P(x)$.
4. Quantifiers:
   - $\forall$: 'for all.'
   - $\exists$: 'there exists.'
   - $\exists!$: 'there exists a unique.'
5. **Examples:**
   (a) 'All elephants are grey.' may be symbolized as: '$\forall\, x : x$ is an elephant $\implies$ x is grey.'
   (b) 'Some elephants are grey.' may be symbolized as: '$\exists\, x : x$ is an elephant $\wedge$ x is grey.'
   (c) 'There is only one grey elephant.' may be symbolized as: $\exists!\, x : x$ is an elephant $\wedge$ x is grey.'
6. Negation of quantified sentences:
   - $\neg\,(\forall\, x : P(x)) \iff (\exists\, x : \neg P(x))$
   - $\neg\,(\exists\, x : P(x)) \iff (\forall\, x : \neg P(x))$
   - $\neg\,(\exists!\, x : P(x)) \iff [(\forall\, x : \neg P(x)) \vee (\exists\, x, y, x \neq y : P(x) \wedge P(y))]$
7. **Examples:**
   - The negation of: 'All elephants are grey.' is: '*Some* elephants are *not* grey.'
   - The negation of: 'Some elephants are grey.' is: '*All* elephants are *not* grey.'
   - The negation of: 'There is only one grey elephant.' is: 'Either *all* elephants are *not* grey, *or* there is *more than one* grey elephant.'
8. Other symbols commonly used:
   - $\in$: 'is an element of' (used to show membership in a set)
   - $\ni$: 'such that.'
   - $\mathbb{N} = \{1, 2, 3, \ldots\}$ a.k.a. 'The natural numbers.'
   - $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ a.k.a. 'The integers.'
   - $\mathbb{Q} = \left\{ \dfrac{x}{y} : x \in \mathbb{Z} \wedge y \in \mathbb{N} \right\}$ a.k.a. 'The rational numbers.'
   - $\mathbb{R}$ a.k.a. 'The real numbers.'
9. Order of quantifiers:
   - '$\forall\, x\, \exists\, y$' permits the choice of y to depend on the choice of x.
   - '$\exists\, y\, \forall\, x$' requires the choice of y work regardless of the choice of x.
10. **Examples:**
    - $\forall\, x \in \mathbb{R}\ \exists\, y \in \mathbb{R}\ \ni x + y = 0$ is **TRUE**: given an x, you can choose $y = -x$.
    - $\exists\, y \in \mathbb{R}\ \forall\, x \in \mathbb{R}\ \ni x + y = 0$ is **FALSE**: because there isn't one y that works for all x.
11. **IMPORTANT EXAMPLE:** $\lim\limits_{x \to a} f(x) = L$ means:

$$\forall\, \epsilon > 0\, \exists\, \delta > 0\ \ni 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$